1 Joint Numerical Range – list of observables

The collection of two-qubit observables X_1, X_2, X_3, such that their joint numerical range $W(X_1, X_2, X_3)$ gives the projection of the 15-dimensional set Ω_4 of density matrices of order four into a 3-space, which gives the blue convex set shown in the Quantum Information section.

The smaller green subset represents the separable numerical range

$$W_{\mathtt{sep}}(X_1, X_2, X_3),$$

which forms a projection of the set of separable states into R^3. Any result of a triple measurement (on three copies of the same state ρ) giving coordinates $x_i = \mathtt{Tr}\rho X_i$ which are outside the green set represent an entangled state. In general the larger distance from the green set of separable states – the larger entanglement.

1) Fig. 1a

$$X_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2) Fig. 1b

It is composed out of product observables only. Let us define a family of $N = 2$ matrices interpolating between σ_x and σ_z, $\sigma(\alpha) = \sin(\alpha)\sigma_x + \cos(\alpha)\sigma_z$. Then

$$X_1 = \sigma_z \otimes \sigma_z = \sigma(0) \otimes \sigma(0)$$
$$X_2 = \sigma(\pi/4) \otimes \sigma(\pi/2)$$
$$X_3 = \sigma(-\pi/4) \otimes \sigma(\pi/2)$$

With a single product observable one cannot detect entanglement. Observe, however, that the green image of the set of separable states determined by three product observables X_1, X_2 and X_3 is much smaller than the blue set being the projection of the entire set of quantum states. A non-zero difference of both sets, $W \setminus W_{\mathtt{sep}}$, directly demonstrates that multiple measurement with a combination of three product observables can be sufficient to detect quantum entanglement. The difference of both sets, $W \setminus W_{\texttt{sep}}$, representing projection of the set of entanglement states is visualized by the model shown in Fig. 2a.

3) Fig. 2a.
Model of the difference of both sets, $W \setminus W_{\texttt{sep}}$, for the same three matrices X_1, X_2 and X_3 as in Fig. 1a.

4) Fig. 2b.
Separable numerical range $W_{\texttt{sep}}(X_1, X_2, X_3)$ with

$$X_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Return the Jagiellonian Quantum Information Team page.