Quantum Sudoku

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What problems can be solved with a system of mutually unbiased bases? How to use symmetry properties to find a distinguished quantum measurement? How do quantum games differ from classical games?

These issues have been addressed by Karol Życzkowski and his collaborators in three papers spanning ten years. The three works on distinguished configurations of quantum states were carried out thanks to the support of the Foundation for Polish Science.

To describe a physical system in quantum theory we use the concept of a quantum state, which is a mathematical tool that allows us to calculate the probability of obtaining a given result of a measurement. In the simplest case, the state is represented by a unit vector constrained to a sphere called the Bloch sphere. A classical system, containing one bit (*binary unit*) of information, can only assume values 0 or 1, which correspond to the poles of the sphere: the north pole N, and the south pole S, respectively. Thus, the corresponding vector indicating the state of the system can only be oriented in the "up" or "down" directions, like a tossed coin that can only fall with heads or tails upward. Another example of a system with two possible states is an electrical conductor: current flows in the conductor or not.

In quantum theory, all superposition states are allowed: the famous Schrödinger's cat can be a little dead and a little alive at the same time. The states of a two-level system – called a qubit (*quantum bit*) – correspond to any point on the sphere. Therefore, the radial vector describing the state of the qubit, imagined by a pencil held by its end, assumes any position on the sphere. Two quantum states are distinguishable when they are orthogonal, corresponding to different classical states, so two pencils held up and downward. The dimension of the space corresponds to the number of distinguishable measurement outcomes and is $n=2$ for a coin toss (two sides of a coin) and $n=6$ for a dice roll (six faces of a dice).

The set of n mutually distinguishable quantum states forms an orthogonal basis. From the viewpoint of quantum mechanics, what is interesting are the constellations of quantum states that define quantum measurements with special symmetry properties. To such a class belong mutually unbiased bases (MUB), which allow one to perform a quantum measurement with a high accuracy. It is not difficult to prove that in an n-dimensional space there exist not more than $n+1$ such bases. The complete set of MUBs for one qubit, $n=2$, consists of three bases represented by three pairs of antipodal points on the sphere, namely such that lie at two opposite poles of that sphere. In the language of the radial vector represented by a spinning pencil, consider $3*2=6$ pencils, such that each pair pointing in two opposite directions corresponds to a single base, and their convex hull gives a regular octahedron spanned by its six vertices – see Fig. 1. In geometry, such a figure has been known since Plato, and in recent years it has also found certain applications in quantum mechanics.

In the 2010 article “On mutually unbiased bases” [1], the authors analyze sets of mutually unbiased bases for higher dimensions, presenting their classification for the cases *n=3,4,5,7*.

Such constellations of states determine quantum measurements with particularly favorable properties. In this widely cited publication, the authors also discuss the most interesting case, *n=6*, for which configurations of only three MUBs are known. Since in this case the upper limit gives *6+1=7* bases, the determination of the maximum number of unbiased bases in this dimension remains an important open problem.

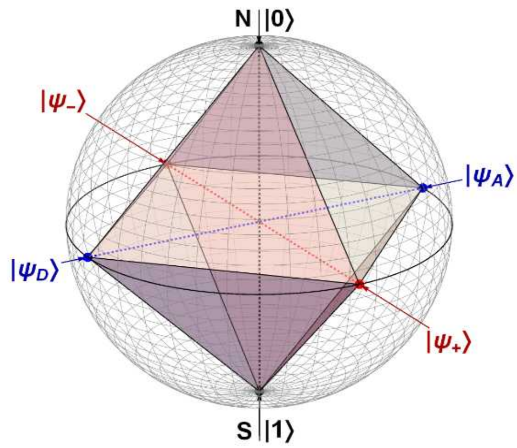

Fig. 1. Three bases consisting of two antipodal points on the sphere form a set of MUBs for dimension n=2

The article “Iso-entangled mutually unbiased bases, symmetric quantum measurements and mixed-state designs” [2] deals with the issue of unbiased bases for the case *n=4*, treated as a system of two qubits. The paper presents a configuration of 20 states in this dimension with the same degree of quantum entanglement, which forms a full set of five unbiased bases. Hence, all these states can be obtained from a particular fiducial state by local transformations, which can facilitate their experimental implementation. The entire system can be represented by another figure by Plato, a regular dodecahedron embedded inside the Bloch sphere. It is convenient to decompose it into five regular tetrahedrons. Each tetrahedron is shown in Fig. 2 in a different color, which corresponds to a single base in four dimensions, which when reduced to two, gives four tetrahedral vertices equidistant from the center of the sphere.

The mutual unbiasedness of the bases is made apparent by the fact that each pair of tetrahedrons intersects symmetrically. Each of the *5*4=20* states that define the entire measurement system were determined analytically. Thus, the authors revealed that this set of 20 states forms a quantum design with special symmetry and special properties. The obtained results are important both from the viewpoint of the foundations of quantum mechanics and the theory of quantum information processing.

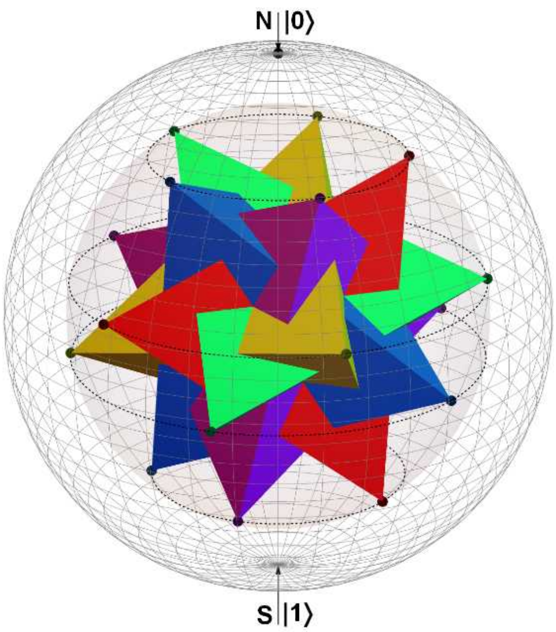

Fig. 2. Five unbiased bases for n=4 with the same degree of quantum entanglement, if reduced to a single subsystem, form five intersecting regular tetrahedrons inscribed into a sphere *(Figure by Jakub Czartowski)*

“In the history of science there have been many prolific ideas about whose application we could say very little at the time of their inception,” says Karol Życzkowski. After all, a significant branch of practical mathematics – the theory of probability – originated from the analysis of multiple coin tosses and games of dice. The key problems for gambling French officers were also interesting for Pascal: Which rules of the game are reasonable and give both parties equal chances to win?

With the development of quantum theory, the question arises how the chances of winning will change if instead of ordinary coins with two sides we use in the game “quantum coins,” which apart from the states 0 = heads and 1 = tails, can also assume superposition states corresponding to any points of the Bloch sphere. At the cross-section of quantum mechanics, probability calculus, and game theory, a new field of science called quantum games has recently emerged. Its goal is to study how a given game changes if players are allowed to use quantum strategies. The related field of quantum finances has a similar status, as it analyzes financial markets under the (currently unrealistic) assumptions that stock market agents can also use quantum strategies and that their portfolios contain superpositions of bought and sold stocks of various companies.

$a)$

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

$b)$

$\lvert 1\rangle$	$\lvert 2\rangle$	$\lvert 3\rangle + \lvert 4\rangle$	$\lvert 3\rangle - \lvert 4\rangle$
$\lvert 3\rangle$	$\lvert 4\rangle$	$\lvert 1\rangle - \lvert 2\rangle$	$\lvert 1\rangle + \lvert 2\rangle$
$\lvert 2\rangle + \lvert 4\rangle$	$\lvert 1\rangle - \lvert 3\rangle$	$\lvert 1\rangle + \lvert 2\rangle + \lvert 3\rangle - \lvert 4\rangle$	$\lvert 1\rangle - \lvert 2\rangle + \lvert 3\rangle + \lvert 4\rangle$
$\lvert 2\rangle - \lvert 4\rangle$	$\lvert 1\rangle + \lvert 3\rangle$	$\lvert 1\rangle + \lvert 2\rangle - \lvert 3\rangle + \lvert 4\rangle$	$\lvert 1\rangle - \lvert 2\rangle - \lvert 3\rangle - \lvert 4\rangle$

Fig. 3. Small examples of sudoku in a 4x4 square: a) classical solution based on the chess-knight motive; b) quantum version of SudoQ with 16 different states forming 4*3=12 different orthogonal bases in dimension n=4. For legibility, quantum states are not normalized.

In a recent article “Genuinely quantum SudoQ and its cardinality” Karol Życzkowski and colleagues describe quantum sudoku patterns [3]. In classical sudoku, every symbol in every row, column, and block of a square must be different. The quantum version of the game, called SudoQ, uses arbitrary quantum states under the assumption that all states in each row, column, and block form an orthogonal basis - see Fig. 3. The presented classification of such arrangements for dimension *n=4* offers original schemes of quantum measurements and may find applications in the development of artificial intelligence and quantum machine learning.

References

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[3] J. Paczos, M. Wierzbiński, G. Rajchel-Mieldzioć, A. Burhardt, K. Życzkowski, Genuinely quantum SudoQ and its cardinality, Phys. Rev. **A 104**, 042423 (2021).